EXAMPLE OF FLOW OF AN AXISYMMETRIC LIQUID STREAM OVER A SELF-PROPELLED BODY

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The plane problem of steady flow of a stream of viscous incompressible liquid over a self-propelled, cylindrical body — a circular cylinder whose moving boundary is a motor — was considered in [1]. In this article we study the axisymmetric problem of steady flow of a stream of viscous, incompressible liquid over a self-propelled sphere. The normal component of the flow velocity vector is distributed over the surface of the sphere so that the mass flux and the total momentum flux of the liquid through this surface equal zero. At low Reynolds numbers, in particular, an asymptotic formula is obtained, according to which the disturbance of flow velocity in the wake behind the body under consideration approaches zero with greater distance by an X^{-2} law, i.e., considerably faster than in the steady axisymmetric wake behind a body imparting a nonzero momentum to the liquid per unit time. In the latter case, as is well known [2], the disturbance of flow velocity approaches zero by an X^{-1} law.

Let X, Y, and Z be rectangular coordinates; a is the radius of the sphere; $x = X/a$, $y = Y/a$, $z = Z/a$; **i**, **j**, **k** are unit vectors, the directions of which coincide with the directions of the x, y, and z axes, respectively; $r = (x^2 + y^2 + z^2)^{1/2}$; θ is the angle between the vectors **i** and $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; λ is the angle between the vectors **j** and $y\mathbf{j} + z\mathbf{k}$; **V** is the flow velocity of the liquid; $\mathbf{V}_\infty = V_\infty \mathbf{i}$ is the flow velocity at infinity ($V_\infty > 0$); $\mathbf{u} = \mathbf{V}/V_\infty$; u_r, r_θ, and u_λ are the r-, θ-, and λ-components, respectively, of the vector **u**; P is the pressure; P_∞ is the pressure at infinity; σ is the density of the liquid; $p = (P - P_\infty)/(\sigma V_\infty^2)$; ν is the kinematic viscosity coefficient; $\mathrm{Re} = aV_\infty/\nu$ is the Reynolds number; ε is a certain dimensionless quantity independent of the coordinates; f is a function of θ defined in the interval $[0, \pi]$; $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

In the notation adopted here, the Navier—Stokes and continuity equations and the conditions which the dimensionless flow velocity and pressure must satisfy have the following form:

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\Delta\mathbf{u}; \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0; \tag{2}$$

$$u_r = \varepsilon f,\ u_\theta = 0,\ u_\lambda = 0 \quad \text{at} \quad r = 1; \tag{3}$$

$$\mathbf{u} \to \mathbf{i},\ p \to 0 \quad \text{as} \quad r \to \infty. \tag{4}$$

The liquid flow is assumed to be symmetric relative to the x axis. This means that u_r, u_θ, and p do not depend on λ and that $u_\lambda \equiv 0$ [in view of which the last of the conditions (3) is satisfied]. Because of this flow symmetry, the y- and z-components of the vector **S** of total momentum flux of the liquid through the surface of the sphere equal zero. Thus, $\mathbf{S} = S\mathbf{i}$. With allowance for the steadiness of this flow, it is easy to show that

$$S/(\sigma a^2 V_\infty^2) = 2\pi r^2 \int_0^\pi \left[(u_r\cos\theta - u_\theta\sin\theta)u_r + (p - 2\mathrm{Re}^{-1}\partial u_r/\partial r)\cos\theta + \right. \tag{5}$$

$$\left. + \mathrm{Re}^{-1}(r^{-1}\partial u_r/\partial\theta + \partial u_\theta/\partial r - r^{-1}u_\theta)\sin\theta\right]\sin\theta d\theta.$$

In accordance with (1)-(5), $S/(\sigma a^2 V_\infty^2)$ is a function of ε and Re for the assigned dependence of f on θ. In this article it is assumed that the distribution of u_r over the sphere r = 1 is such that

$$S = 0, \tag{6}$$

and that ε is defined as a function of Re by Eq. (6) (for the assigned dependence of f on θ). Moreover, it is assumed that the mass flux of liquid through the surface of the sphere equals zero. In accordance with this, the function f(θ) satisfies the condition

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$$\int_0^\pi f(\theta)\sin\theta d\theta = 0. \tag{7}$$

The problem (1)-(4) is analyzed below for low Reynolds numbers.

We assume that as Re → 0

$$\mathbf{u}(r, \theta, \mathrm{Re}) \approx \mathbf{u}_0(r, \theta) + \sum_{m=1}^{\infty} \mathbf{u}_m(r, \theta)\, g_m(\mathrm{Re}); \tag{8}$$

$$p(r, \theta, \mathrm{Re}) \approx p_0(r, \theta)\,\mathrm{Re}^{-1} + \sum_{m=1}^{\infty} p_m(r, \theta)\,\mathrm{Re}^{-1} g_m(\mathrm{Re}), \tag{9}$$

where $g_m(\mathrm{Re})$ are functions satisfying the conditions

$$\lim_{\mathrm{Re}\to 0} g_1 = 0, \quad \lim_{\mathrm{Re}\to 0} \frac{g_{m+1}}{g_m} = 0.$$

We note that the λ-components of the vectors $\mathbf{u}_n$ (n = 0, 1, ...) equal zero. Asymptotic expansions obtained as Re → 0 and for constant r and θ will be called internal. We expand the function ε(Re) in the following asymptotic series as Re → 0:

$$\varepsilon(\mathrm{Re}) \approx \varepsilon_0 + \sum_{m=1}^{\infty} \varepsilon_m g_m(\mathrm{Re}). \tag{10}$$

Using (1)-(4) and (8)-(10), we define the zeroth-approximation problem:

$$\nabla p_0 = \Delta \mathbf{u}_0; \tag{11}$$

$$\nabla\cdot\mathbf{u}_0 = 0; \tag{12}$$

$$u_{0r} = \varepsilon_0 f,\ u_{0\theta} = 0 \quad \text{at} \quad r = 1; \tag{13}$$

$$u_{0r} \to \cos\theta,\ u_{0\theta} \to -\sin\theta,\ p_0 \to 0 \quad \text{as} \quad r \to \infty, \tag{14}$$

where u_{0r} and $u_{0\theta}$ are the r- and θ-components, respectively, of the vector $\mathbf{u}_0$.

We assume that

$$f = \sum_{m=0}^{\infty} f_m P_m(\cos\theta), \tag{15}$$

where f_m are constants, $f_1 \neq 0$; P_m are Legendre polynomials. In accordance with the condition (7), we will have

$$f_0 = 0. \tag{16}$$

In Eq. (5) for $S/(\sigma a^2 V_\infty^2)$ we replace u_r, u_θ, and p by their internal expansions. Using the expression obtained as a result, we find

$$S/(\sigma a^2 V_\infty^2) \approx s_{-1}\mathrm{Re}^{-1} + s_0 + s_0'\,\mathrm{Re}^{-1} g_1(\mathrm{Re}) + \ldots \quad \text{as} \quad \mathrm{Re} \to 0. \tag{17}$$

Here, in particular,

$$s_{-1} = 2\pi r^2 \int_0^\pi \left[(p_0 - 2\partial u_{0r}/\partial r)\cos\theta + (r^{-1}\partial u_{0r}/\partial\theta + \partial u_{0\theta}/\partial r - r^{-1}u_{0\theta})\sin\theta\right]\sin\theta d\theta.$$

In view of the fact that S = 0, all terms of the expansion (17) must also equal zero. Using (11)-(16) and the condition that the leading term of the expansion (17) equals zero, and having made simple calculations, we obtain

$$\varepsilon_0 = 3/f_1; \tag{18}$$

$$p_0 = \frac{3}{f_1} \sum_{m=2}^{\infty} \frac{m(2m-1)}{m+1} f_m r^{-m-1} P_m(\cos\theta); \tag{19}$$

$$u_{0r} = \frac{1}{r^2 \sin\theta} \frac{\partial \psi_0}{\partial \theta}, \quad u_{0\theta} = -\frac{1}{r \sin\theta} \frac{\partial \psi_0}{\partial r}, \tag{20}$$

where $\psi_0 = -(r^2 + 2r^{-1}) \int\limits_{-1}^{\cos\theta} P_1(\xi)\, d\xi - \frac{3}{2f_1} \sum\limits_{m=2}^{\infty} f_m [m r^{2-m} + (2-m) r^{-m}] \int\limits_{-1}^{\cos\theta} P_m(\xi)\, d\xi.$

The region of applicability of the solution (19), (20) is determined by the condition of smallness of convective terms compared with viscosity terms in the equations of motion of the liquid. Estimating their values using (20), we can show that this condition is not satisfied for Re r $\gtrsim$ 1. Along with the internal expansions (8) and (9), therefore, we also consider certain external expansions of **u** and p below. The terms of these expansions match, in a definite way (in accordance with the principle of asymptotic joining [3]), with the terms of the expansions (8) and (9). We note that the conditions (14) coincide with the conditions for matching the leading terms of the internal expansions of u_r, u_θ, and p with the terms of order unity in their external expansions.

We rewrite Eqs. (1) and (2) and the conditions (3) and (4) for u_r, u_θ, and p as follows:

$$(\mathbf{u} \cdot \hat{\nabla})\mathbf{u} = -\hat{\nabla} p + \hat{\Delta}\mathbf{u}; \tag{21}$$

$$\hat{\nabla} \cdot \mathbf{u} = 0; \tag{22}$$

$$u_r = \varepsilon f, \; u_\theta = 0 \quad \text{for} \quad \rho = \text{Re}; \tag{23}$$

$$u_r \to \cos\theta, \; u_\theta \to -\sin\theta, \; p \to 0 \quad \text{as} \quad \rho \to \infty. \tag{24}$$

Here $\hat{\nabla} = (\partial/\partial\hat{x}, \partial/\partial\hat{y}, \partial/\partial\hat{z})$; $\hat{\Delta} = \partial^2/\partial\hat{x}^2 + \partial^2/\partial\hat{y}^2 + \partial^2/\partial\hat{z}^2$; $\hat{x}$ = Re x, $\hat{y}$ = Re y, $\hat{z}$ = Re z; ρ = Re r. We assume that as Re $\to$ 0,

$$\mathbf{u}(\rho/\text{Re}, \theta, \text{Re}) \approx \mathbf{i} + \sum_{m=1}^{\infty} \mathbf{u}^{(m)}(\rho, \theta)\, h_m(\text{Re}); \tag{25}$$

$$p(\rho/\text{Re}, \theta, \text{Re}) \approx \sum_{m=1}^{\infty} p^{(m)}(\rho, \theta)\, h_m(\text{Re}), \tag{26}$$

where h_m(Re) are functions satisfying the conditions

$$\lim_{\text{Re}\to 0} h_1 = 0, \; \lim_{\text{Re}\to 0} \frac{h_{m+1}}{h_m} = 0.$$

We note that the λ-components of the vectors $\mathbf{u}^{(m)}$ equal zero. Asymptotic expansions obtained for Re $\to$ 0 and constant ρ and θ will be called external. Using (21), (22), and (24)-(26), we obtain

$$\frac{\partial \mathbf{u}^{(1)}}{\partial \hat{x}} = -\hat{\nabla} p^{(1)} + \hat{\Delta}\mathbf{u}^{(1)}; \tag{27}$$

$$\hat{\nabla} \cdot \mathbf{u}^{(1)} = 0; \tag{28}$$

$$u_r^{(1)} \to 0, \; u_\theta^{(1)} \to 0, \; p^{(1)} \to 0 \quad \text{as} \quad \rho \to \infty, \tag{29}$$

where $u_r^{(1)}$ and $u_\theta^{(1)}$ are the r- and θ-components, respectively, of the vector $\mathbf{u}^{(1)}$. We note that the boundary conditions (23) cannot be used to obtain the conditions which the r- and θ-components of the vectors $\mathbf{u}^{(m)}$ (m = 1, 2, ...) must satisfy, since the external expansions are carried out with constant ρ and θ and Re $\to$ 0, while ρ = Re in (23), and therefore ρ cannot remain constant as Re $\to$ 0.

Equations (27) and (28) have solutions of the type [4]

$$u_r^{(1)} = \frac{\partial \varphi}{\partial \rho} + \frac{\partial \chi}{\partial \rho} - \chi \cos\theta; \tag{30}$$

$$u_\theta^{(1)} = \rho^{-1} \frac{\partial \varphi}{\partial \theta} + \rho^{-1} \frac{\partial \chi}{\partial \theta} + \chi \sin\theta; \tag{31}$$

$$p^{(1)} = \rho^{-1}\frac{\partial\varphi}{\partial\theta}\sin\theta - \frac{\partial\varphi}{\partial\rho}\cos\theta, \tag{32}$$

where $\varphi(\rho, \theta)$, $\chi(\rho, \theta)$ are functions satisfying the equations

$$\hat{\Delta}\varphi = 0; \tag{33}$$

$$\partial\chi/\partial\hat{x} = \hat{\Delta}\chi. \tag{34}$$

Solving Eqs. (33) and (34) and using (29)-(32), we obtain

$$\varphi = \sum_{m=0}^{\infty} a_m \rho^{-m-1} P_m(\cos\theta); \tag{35}$$

$$\chi = \rho^{-\frac{1}{2}} e^{\frac{1}{2}\rho\cos\theta} \sum_{m=0}^{\infty} A_m K_{m+\frac{1}{2}}(\rho/2) P_m(\cos\theta), \tag{36}$$

where a_m and A_m are constants; $K_{m+1/2}$ are MacDonald functions.

Let the coefficient f_2 in the expansion (15) differ from zero. Then the external expansions of $\mathbf{u}_0 - \mathbf{i}$ and $\mathrm{Re}^{-1}p_0$ start with terms of order Re^2. In accordance with this, we set $h_1(\mathrm{Re}) = \mathrm{Re}^2$. We replace r by ρ/Re in Eq. (5) for $S/(\sigma a^2 V_\infty^2)$. Using the resulting expression and (25) and (26), we find

$$S/(\sigma a^2 V_\infty^2) \approx \hat{s}_0 + \hat{s}_1 \mathrm{Re}^2 + \hat{s}_1' \mathrm{Re}^{-2} h_2(\mathrm{Re}) + \ldots \quad \text{as} \quad \mathrm{Re} \to 0. \tag{37}$$

Here, in particular,

$$\hat{s}_0 = 2\pi\rho^2 \int_0^{\pi} \Big[\tfrac{1}{2} u_r^{(1)}(3 + \cos 2\theta) - \tfrac{1}{2} u_\theta^{(1)} \sin 2\theta + \left(p^{(1)} - 2\partial u_r^{(1)}/\partial\rho\right)\cos\theta +$$
$$+ \left(\rho^{-1}\partial u_r^{(1)}/\partial\theta + \partial u_\theta^{(1)}/\partial\rho - \rho^{-1}u_\theta^{(1)}\right)\sin\theta\Big] \sin\theta d\theta.$$

Since S = 0, all terms of the expansion (37) must also equal zero.

To find $u_r^{(1)}$, $u_\theta^{(1)}$, and $p^{(1)}$ we must determine the constants a_m and A_m (m = 0, 1, ...) in Eqs. (35) and (36) for φ and χ. These constants must satisfy the condition that the leading term of the expansion (37) equal zero and the matching conditions

$$E_{\mathrm{Re}^2} I_1 u_r = I_1 E_{\mathrm{Re}^2} u_r, \quad E_{\mathrm{Re}^2} I_1 u_\theta = I_1 E_{\mathrm{Re}^2} u_\theta,$$
$$E_{\mathrm{Re}^2} I_{\mathrm{Re}^{-1}} p = I_{\mathrm{Re}^{-1}} E_{\mathrm{Re}^2} p,$$

where $I_{\mathrm{Re}^{-1}}$, I_1, and E_{Re^2} are operators of the internal and external expansions, defined as given in [1]. The enumerated conditions are satisfied for

$$\begin{gathered} a_0 = 0, \ a_1 = 3f_2/f_1, \\ A_0 = 3f_2/(2\pi^{1/2}f_1), \ A_1 = -3f_2/(2\pi^{1/2}f_1), \\ a_k = 0, \ A_k = 0 \ (k = 2, 3, \ldots). \end{gathered} \tag{38}$$

Thus, in accordance with (30)-(32), (35), (36), and (38), $u_r^{(1)}$, $r_\theta^{(1)}$, and $p^{(1)}$ have the form

$$u_r^{(1)} = \frac{3f_2}{8f_1\rho} e^{\frac{1}{2}\rho(\cos\theta - 1)} \{\cos 2\theta - 1 + 2\rho^{-1}(\cos 2\theta + 4\cos\theta - 1) + 16\rho^{-2}\cos\theta\} - \frac{6f_2}{f_1\rho^3}\cos\theta,$$

$$u_\theta^{(1)} = \frac{3f_2}{4f_1\rho} e^{\frac{1}{2}\rho(\cos\theta - 1)} \{(1 + 2\rho^{-1})(1 - \cos\theta) + 4\rho^{-2}\}\sin\theta - \frac{3f_2}{f_1\rho^3}\sin\theta, \quad p^{(1)} = \frac{3f_2}{2f_1\rho^3}(3\cos 2\theta + 1).$$

Applying the method of additive composition [3], we find approximate composite expressions for u_r, u_θ, and p applicable in the entire region of flow:

$$u_r \approx (I_1 + E_{\mathrm{Re}^2} - I_1 E_{\mathrm{Re}^2}) u_r = u_{0r} + \mathrm{Re}^2 u_r^{(1)} - \frac{3f_2}{4f_1 r^2}(3\cos 2\theta + 1),$$
$$u_\theta \approx (I_1 + E_{\mathrm{Re}^2} - I_1 E_{\mathrm{Re}^2}) u_\theta = u_{0\theta} + \mathrm{Re}^2 u_\theta^{(1)},$$
$$p \approx (I_{\mathrm{Re}^{-1}} + E_{\mathrm{Re}^2} - I_{\mathrm{Re}^{-1}} E_{\mathrm{Re}^2}) p = \mathrm{Re}^{-1} p_0.$$

Let us consider the question of the asymptotic behavior of the flow velocity at large distances from the sphere (at low Re). In the composite expressions for u_r and u_θ we convert to the coordinates ρ and θ. For any positive number δ not exceeding π as $\rho \to \infty$, $\delta \leqslant \theta \leqslant \pi$ and constant Re we will have

$$u_r \sim \cos\theta + O(\rho^{-3}),\ u_\theta \approx -\sin\theta + O(\rho^{-3}).$$

The disturbance in flow velocity approaches zero considerably more slowly as the distance increases in the wake behind the body on paraboloids $(\hat{y}^2 + \hat{z}^2)/\hat{x} = \text{const}$ $(\hat{x} > 0)$. We convert to the coordinates $\hat{x}$ and $(\hat{y}^2 + \hat{z}^2)/\hat{x}$ in the composite expressions for u_r and u_θ. We expand the resulting expressions as $\hat{x} \to +\infty$ and for constant $(\hat{y}^2 + \hat{z}^2)/\hat{x}$ and Re to within terms small compared with $\hat{x}^{-2}$. Then converting to dimensional quantities and the vector form of notation, we find

$$\mathbf{V} \approx \mathbf{V}_\infty\left\{1 + \frac{3f_2a^2}{f_1X^2}\left(1 - V_\infty\frac{Y^2+Z^2}{4\nu X}\right)e^{-V_\infty\frac{Y^2+Z^2}{4\nu X}}\right\}.$$

Thus, at low Reynolds numbers the disturbance in flow velocity in the wake behind the body under consideration tends toward zero by an X^{-2} law as the distance increases [the distance from the body coincides with X to within a quantity small compared with X as $X/a \to +\infty$ for constant $(Y^2 + Z^2)/(aX)$)]. This result, in particular, is in agreement with a comment made in [5] about the law of variation with distance of the disturbance in the velocity of axisymmetric flow in the wake behind a self-propelled body.

LITERATURE CITED

1. V. L. Sennitskii, "Liquid flow around a self-propelled body," Zh. Prikl. Mekh. Tekh. Fiz., No. 3 (1978).
2. L. D. Landau and E. M. Lifshits, Mechanics of Continuous Media [in Russian], Gos. Izd. Tekh. Teor. Lit., Moscow (1954).
3. M. Van Dyke, Perturbation Methods in Fluid Mechanics, Academic Press, New York (1964).
4. N. E. Kochin, I. A. Kibel', and N. V. Roze, Theoretical Fluid Mechanics [in Russian], Part 2, Fizmatgiz, Moscow (1963).
5. G. Birkhof and E. H. Zarantonello, Jets, Wakes, and Cavities, Academic Press, New York (1957).